Notes on Orientation and Integration

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1 Orientation Part I

In this section, we define orientable manifold using the concept of frames.

Definition 1.1 (Orientation of $\mathbb{R}^n$)
Two ordered bases of $\mathbb{R}^n$ are said to be equivalent if the change-of-basis matrix has positive determinant. An orientation of $\mathbb{R}^n$ is an equivalence class of ordered bases.

To define orientation for general manifold, intuitively, we want to orient the tangent space at each point in a "coherent" manner.

Definition 1.2 (Frame)
A frame for the tangent bundle $TM \to M$ over an open set U is a collection of nowhere-vanishing (possibly discontinuous) vector fields $(X_1, \cdots, X_n)$, such that $(X_{1,p}, \cdots, X_{n,p})$ forms a basis of T_pM for every $p \in U$.

The definition of frame can be generalised to any vector bundle, but we focus on tangent bundle for now. We say two frames are equivalent if the change-of-basis matrix has positive determinant at every point in M. Now define **pointwise orientation** μ to be the assignment of orientation μ_p to each T_pM. In other words, μ is the equivalent class of a (possibly discontinuous) global frame. We say μ is continuous at p if p has an open neighbourhood U such that there exist a continuous frame $(X_1, \cdots, X_n)$, which satisfies $\mu_q = [(X_{1,q}, \cdots, X_{n,q})]$ for every $q \in U$. An **orientation** of M is an pointwise orientation that is continuous everywhere. A manifold is said to be **orientable** if it has an orientation. A manifold together with an orientation is said to be **oriented**.

Proposition 1.3
A connected orientable manifold M has exactly two orientations.

Proof: Let μ and ν be two orientations on M. Define function $f : M \to \{\pm 1\}$ by

$$f(p) = \begin{cases} 1 & \mu_p = \nu_p \\ -1 & \mu_p \neq \nu_p \end{cases}$$

For any $p \in M$, there is connected neighbourhood U such that $\mu_q = [(X_{1,q}, \cdots, X_{n,q})]$, $\nu_q = [(Y_{1,q}, \cdots, Y_{n,q})]$. The determinant of change-of-basis matrix is a continuous function, so it is everywhere-positive or everywhere-negative, so f is constant on U. It is easy to prove that a locally constant function on a connected set is constant, so the proposition is proved. ∎

2 Orientation Part II

In this section, we characterise orientation by nowhere-vanishing top forms, which turns out to be convenient in many cases.

We first discuss the pullback of differential forms. Suppose $F : N \to M$ is a C^∞ map of manifolds. There is differential at $p \in N$

$$F_{*,p} : T_pN \to T_pM.$$

Now define pullback map $F^*_{*,p} : \bigwedge^k T_pM \to \bigwedge^k T_pN$ of k-covector at p by

$$F^*_p(f)(v_1, \cdots, v_k) = f(F_{*,p}(v_1), \cdots, F_{*,p}(v_k)), \quad v_i \in T_pN.$$

This is a generalization of dual map. Define pullback map $F^*: \Omega^k(M) \to \Omega^k(N)$ of differential k-form point-wise:

$$(F^*(\omega))_p = F_p^*(\omega_p).$$

Pullback map is $\mathbb{R}$-linear, respects wedge product, and commutes with exterior derivative (hence it is a chain map between de Rham complexes). Also notice covector fields have pullback while vector fields generally cannot be push forward. As a result, information can be more easily transferred between manifolds in the form of covector fields.

Proposition 2.1

Given smooth map $F: N \to M$, point $p \in N$, $(U, x^1, \cdots, x^n)$ and $(V, y^1, \cdots, y^m)$ be coordinate charts about $p \in N$ and $F(p) \in M$ respectively. Then the pullback of dy^i

$$F_p^*(dy^i) = \sum_j \frac{\partial F^i}{\partial x^j} dx^j = dF^i$$

where $F^i = y^i \circ F$.

Proof: Skipped. ■

An immediate consequence is that when $m = n$, $F_p^*(dy^1 \wedge \cdots \wedge dy^n) = JF(p)\ dx^1 \wedge \cdots \wedge dx^n$.

Now we give an equivalent definition of orientable manifold. An atlas $\{(U_\alpha, \phi_\alpha)\}$ of M is said to be **oriented** if all transition functions $g_\alpha\beta = \phi_\alpha \phi_\beta^{-1}$ are orientation-preserving, that is, their Jacobian determinant are everywhere positive. An **oriented manifold** is a manifold with oriented atlas.

Proposition 2.2

M is orientable if and only if it has an oriented atlas.

Proof: The proof can be found on [Tu] (Theorem 21.10). The idea is to show that a continuous pointwise orientation induces an oriented atlas, and vice versa. ■

Proposition 2.3

An n-manifold M is orientable if and only if it has a global nowhere vanishing n-form.

Proof: By Proposition 2.1, $T: \mathbb{R}^n \to \mathbb{R}^n$ is orientation-preserving if and only if $T^* dx^1 \wedge \cdots \wedge dx^n$ is a positive multiple of $dx^1 \wedge \cdots \wedge dx^n$.

($\Leftarrow$): Let ω be the nowhere vanishing n-form. Let $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ be a chart, $dx^1 \wedge \cdots \wedge dx^n$ be n-form on $\mathbb{R}^n$. Then since $\phi_\alpha^* dx^1 \wedge \cdots \wedge dx^n$ is nowhere vanishing, there exist nowhere vanishing real valued function f_α on U_α such that $\phi_\alpha^* dx^1 \wedge \cdots \wedge dx^n = f_\alpha \omega$. Apparently f_α is ether everywhere positive or everywhere negative. "Flip" all charts with negative f_α to ensure that every chart has positive f_α. Now that every chart is in the same "direction" with ω, it is easy to verify that $(\phi_\alpha \circ \phi_\beta^{-1})^*$ pulls $dx^1 \wedge \cdots \wedge dx^n$ to its positive multiple. So every transition function is orientation-preserving.

($\Rightarrow$): Use partition of unity to piece together $\phi_\alpha^* dx^1 \wedge \cdots \wedge dx^n$. ■

Any two global nowhere vanishing n-forms $\omega$and ω' on an orientable manifold M of dimension n differ by a nowhere vanishing function: $\omega = f\omega'$. If M is connected, then f is ether everywhere positive or everywhere negative. We say that ω and ω' are equivalent if f is positive. Either class is called an **orientation** on M, written $[M]$.

We now discuss orientation of manifold with boundary. A manifold M of dimension n with boundary is given by an atlas $\{(U_\alpha, \phi_\alpha)\}$, where U_α is homeomorphic to ether $\mathbb{R}^n$ or upper half space $\mathbb{H}^n$. The boundary of M is an orientable $(n-1)$-manifold, as a result of the following lemma.

Lemma 2.4

Let $T: \mathbb{H}^n \to \mathbb{H}^n$ be a diffeomorphism of the upper half space with everywhere positive Jacobian determinant. T induces a diffeomorphism $\bar{T}$ of $\partial\mathbb{H}^n \approx \mathbb{R}^{n-1}$ to itself, which has positive Jacobian determinant everywhere.

Proof: Lemma 3.4 on GTA82. ■

Given standard orientation $dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{H}^n$, the **induced orientation** on $\partial\mathbb{H}^n$ is defined by $(-1)^n dx^1 \wedge \cdots \wedge dx^{n-1}$ for $n > 1$, and -1 for $n = 1$. The **induced orientation** on ∂M is defined by the pullback of induced orientation on $\partial\mathbb{H}^n$ by coordinate chart.

3 Integration

The integration of top form of $\mathbb{R}^n$ is defined in the usual sense. In general, let M be an orientable manifold, choose an orientation $[M]$. Given a top form $\tau \in \Omega_c^n(M)$, we define its integral by partitioning τ with respect to the oriented atlas, integrating individually the pullback of each component, and then taking the sum. Namely,

$$\int_M \tau = \sum_\alpha \int_{U_\alpha} \rho_\alpha \tau = \sum_\alpha \int_{\phi_\alpha(U_\alpha)} (\phi_\alpha^{-1})^*(\rho_\alpha \tau)$$

This is in fact well-defined.

Proposition 3.1

The definition of the integral is independent of the oriented atlas and the partition of unity.

Proof: We first observe that when T is a diffeomorphism from $V \subseteq \mathbb{R}^n$ to $U \subseteq \mathbb{R}^n$, we have

$$\int_U f \ dx^1 \wedge \cdots \wedge dx^n = \int_V f \circ T \ |J(T)| \ dy^1 \wedge \cdots \wedge dy^n.$$

By Proposition 2.1, we have $\int_V T^*\omega = \pm \int_U \omega$ where the sign depends on whether T is orientation-preserving.

Now we get to the proof of the proposition. Let $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ be two oriented atlas of the same orientation, ρ_α and χ_β are partitions of unity subordinate to $\{U_\alpha\}$ and $\{V_\beta\}$ respectively. We now have

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \tau = \sum_{\alpha,\beta} \int_{U_\alpha} \rho_\alpha \chi_\beta \tau.$$

Notice $\rho_\alpha \chi_\beta \tau$ has support in $S_{\alpha\beta} = U_\alpha \cap V_\beta$. Now use the observation, we have

$$\int_{U_\alpha} \rho_\alpha \chi_\beta \tau = \int_{\phi_\alpha(S_{\alpha\beta})} (\phi_\alpha^{-1})^*(\rho_\alpha \chi_\beta \tau) = \int_{\psi_\beta(S_{\alpha\beta})} (\phi_\alpha \circ \psi_\beta^{-1})^* \left((\phi_\alpha^{-1})^*(\rho_\alpha \chi_\beta \tau)\right) = \int_{V_\beta} \rho_\alpha \chi_\beta \tau.$$

So $\sum_\alpha \int_{U_\alpha} \rho_\alpha \tau = \sum_{\alpha,\beta} \int_{V_\beta} \rho_\alpha \chi_\beta \tau = \sum_\beta \int_{V_\beta} \chi_\beta \tau.$ ■

The observation in the proof of Proposition 3.1 can be easily generalized. Let $F : N \to M$ be a diffeomorphism, then

$$\int_N F^*\tau = \pm \int_M \tau.$$

We now prove the Stokes' Theorem.

Theorem 3.2 (Stokes' Theorem)

If ω is an $(n-1)$-form with compact support on an oriented n-manifold M, and if ∂M is given the induced orientation, then

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof: It is easy to prove Stokes' theorem for $\mathbb{R}^n$ and $\mathbb{H}^n$ by direct calculation. For the general case, let $\{U_\alpha\}$ be an oriented atlas. By linearity, we only need to show that for each α, there is

$$\int_M d \ \rho_\alpha \omega = \int_{\partial M} \rho_\alpha \omega.$$

To prove this, notice that $\rho_\alpha \omega$ has compact support in U_α, so we can shrink the domain of integration. Now we only need to show that

$$\int_{U_\alpha} d\ \rho_\alpha \omega = \int_{\partial U_\alpha} \rho_\alpha \omega.$$

This is obvious since U_α is diffeomorphic to $\mathbb{R}^n$ or $\mathbb{H}^n$, and we already know that Stokes' theorem is true for $\mathbb{R}^n$ and $\mathbb{H}^n$. ■

To end this section, we mention a useful theorem which makes computations easier. This theorem allow us to "chop up" the manifold into a finite number of pieces, and compute the integral on each piece separately by means of local parametrizations.

Proposition 3.3 (Integration Over Parametrizations)
Let M be an oriented smooth n-manifold (with or without boundary), and let ω be a compactly supported n-form on M. $D_1, \cdots, D_k$ are open domains of integration (bounded subset whose boundary has measure zero) in $\mathbb{R}^n$, and for $i = 1, \cdots, k$. Given smooth maps $F_i : \bar{D}_i \to M$ and let $W_i = F_i(D_i)$, if

(i) $F_i|_{D_i}$ are orientation-preserving diffeomorphisms;
(ii) W_i do not intersect each other;
(iii) $supp\ \omega \subseteq \cup \bar{D}_i$;

then

$$\int \omega = \sum_{i=1}^{k} \int_{D_i} F_i^* \omega.$$

Proof: By linearity of the definition of integration, it suffices to prove the proposition for ω with its support inside a single chart. Now pullback the integration through the chart to $\mathbb{R}^n$, and recall the properties of integrals on $\mathbb{R}^n$. Details can be found on GTA218 (Proposition 16.8). ■